Vector space (सदिश समष्टि)

Let $(F, +, \cdot)$ be a field and $V \neq \phi$ then V is said to be vector space over the field F if two binary composition defined in V i.e. vector addition $+ : V \times V \to V$ and scalar multiplication $\cdot : F \times V \to V$ satisfy following properties.

(i) $(V, +)$ is an abelian group.

(ii) Scalar multiplication is distributive over vector addition.

$$a \cdot (\alpha + \beta) = a \cdot \alpha + a \cdot \beta, \ \forall a \in F \ \& \ \forall \alpha, \beta \in V$$

(iii) Distributivity over scalar addition

$$(a+b) \cdot \alpha = a \cdot \alpha + b \cdot \alpha, \ \forall a, b \in F \ \& \ \forall \alpha \in V$$

(iv) Associativity over scalar multiplication.

$$(ab) \cdot \alpha = a(b \cdot \alpha), \ \forall a, b \in F \ \& \ \forall \alpha \in V$$

(v) $$1 \cdot \alpha = \alpha, \ \forall \alpha \in V$$

where 1 is the <u>unit</u> <u>Element</u> of field F.

Notation of vector space is $\underbrace{V(F)}$

vector space over the field F.

Elements of V are $\rightarrow$ vectors

Elements of F are $\rightarrow$ Scalars

Trivial space or Null space :$\rightarrow$ (शून्य समष्टि)

A vector space which contain only zero vector is known as Trivial space or Null space.

Ex:- The set of positive real numbers R^+ is a vector space over the field R of real numbers for the operation vector addition '$\oplus$' and scalar multiplication '$\odot$' as

$$x \oplus y = xy \quad \forall\, x, y \in R^+$$
$$a \odot x = x^a \quad \forall\, x \in R^+, \forall\, a \in R.$$

Sol^n:$\rightarrow$ given that

$$x \oplus y = xy \quad \forall\, x, y \in R^+ \text{ --- (i)}$$
$$a \odot x = x^a \quad \forall\, x \in R^+, \forall\, a \in R \text{ --- (ii)}$$

Since ~~additi~~ multiplication of two positive real number is again a positive real number so

$$x, y \in R^+ \Rightarrow x \oplus y = xy \in R^+$$

$\Rightarrow$ R^+ is closed w.r. to vector addition.

or vector addition in well defined in R^+.

For $a \in R$, $x \in R^+ \Rightarrow a \odot x = x^a \in R^+$

(as every real power of a positive real number is again a positive real number),

$\Rightarrow R^+$ is closed w.r. to scalar multiplication.

(i) $(R^+, \oplus)$ is an abelian group.

(a) Associativity property:- let $x, y, z \in R^+$

then $x \oplus (y \oplus z) = x \oplus (yz)$ (by (i))

$= x(yz)$ (by (i))

$= (xy)z$ (real no. holds associative prop.)

$= (xy) \oplus z$ (by (i))

$= (x \oplus y) \oplus z$ (by (i))

$\Rightarrow$ Associativity holds in R^+.

(b) Existence of identity:→

Since $\forall\ x \in R^+\ \exists\ 1 \in R^+$ s.t.

$1 \oplus x = x \oplus 1 = x.1$ (by (i))

$= x$

$\Rightarrow 1$ is the additive identity element in R^+.

(c) Existence of inverse:→

$\forall\ x \in R^+\ \exists\ \frac{1}{x} \in R^+$ s.t.

$x \oplus \frac{1}{x} = \frac{1}{x} \oplus x = x\left(\frac{1}{x}\right) = 1$ (identity element)

$\Rightarrow$ Every element of R^+ has its additive inverse in R^+

(iv) commutative property :- Let $x, y \in R^+$

$x \oplus y = xy$ (by (i))

$= yx$ (multiplication of real no. holds commutative prop.)

$= y \oplus x$

$\Rightarrow$ commutative property holds in R^+ for addition.

So by (i), (ii), (iii) & (iv)

$\Rightarrow$ $(R^+, \oplus)$ is an abelian group.

(ii) Scalar multiplication is distributive over vector addition:-

Let $x, y \in R^+$ & $a \in R$, then

$a \odot (x \oplus y) = a \odot (xy)$ (by (i))

$= (xy)^a$ (by (ii))

$= (x^a)(y^a)$

$= x^a \oplus y^a$ (by (i))

$= (a \odot x) \oplus (a \odot y)$ (by (ii))

$\Rightarrow$ scalar multiplication is distributive over vector addition

(iii) distributivity over scalar ~~multipli~~-addition:-

let $a, b \in R$ & $x \in R^+$, then

$$(a+b) \odot x = x^{(a+b)} \quad (\text{by (ii)})$$
$$= x^a . x^b$$
$$= x^a \oplus x^b \quad (\text{by (i)})$$
$$= (a \odot x) \oplus (b \odot x) \quad (\text{by (ii)})$$

$\Rightarrow$ vector ~~addition~~ ~~is~~ multiplication is distributive over scalar addition in R^+

(iv) Associativity over scalar multiplication:-

Let $a, b \in R$ & $x \in R^+$, then

$$a \odot (b \odot x) = a \odot (x^b) \quad (\text{by (ii)})$$
$$= (x^b)^a \quad (\text{by (ii)})$$
$$= x^{ba}$$
$$= x^{ab} \quad [\because a, b \in R \Rightarrow ab = ba]$$
$$= (ab) \odot x \quad (\text{by (ii)})$$

$\Rightarrow$ R^+ holds Associativity over scalar multiplication.

(v) Since 1 is the unit element in R.

& $\forall x \in R^+$ we have

$$1 \odot x = x^1 = x$$

$\Rightarrow$ R^+ satisfy all the properties (i), (ii), (iii), (iv), (v)) of vector space

$\Rightarrow$ $R^+ (R)$ is a vector space.